Comparing the Upwind and Central Methods in Estimating Bed Slope Source Term in SWE With Standing Shock Waves

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Abstract

In this paper, depth averaged shallow water equations are solved numerically in a supercritical flow with standing shock waves. The algorithm applying finite volume Roe-TVD method on unstructured triangular cells. Bed fraction and turbulence source terms are computed with manning formula and ASM models respectively. Ordinary center and an upwind scheme are used for considering bed slope source term. Flow in canal with a side baffle are simulated numerically and results compared with experimental data. Numerical results show that the Roe-TVD scheme can simulated standing shocks carefully and the central method for bed slope source term have better results than upwind method.

Key words: Roe-TVD, Shallow Water Equations, Bed Slope Source Term, ASM, Oblique Waves

1. Introduction

Free surface flow is a ubiquitous physical phenomenon of practical interest to many scientists and engineers. For example, flows such as ocean tides, wind waves, dam break, river floods, and tsunamis support a large interest from diverse fields for diverse reasons.

The general approach to simulate the free surface flow is to compute the fully 3D flow with considering a specific boundary conditions for the free surface. This method most often not used. The main drawback to a fully 3D approach in environmental problems is it's computational cost; because in this situations the size of the spatial domain is very large. For this reason, it is not yet efficient to use the fully 3D approach in river engineering applications and its more Appropriate to use 2D equations such as depth averaged Shallow water equations(SWE), Duan (2004) and Minh-Duc et al. (2004).

Depth averaged Shallow water equations are obtained with some assumptions from depth integration of Navier–Stokes equations. Many of natural flows such as flow in rivers can be simulated by these equations.

The numerical solving of the SWE has been the subject of extensive literature. the most commonly chosen numerical method is Finite Volumes method(FVM). Applying the finite volume method, the integral forms of the conservation laws are discretized over the computational domain into small structured/unstructured cells. Here, the main objective is to

find the normal fluxes at the faces of the cells, Toro (2001). The most common stabilisation methods for calculating numerical flux are the upwind schemes and the centred schemes with artificial diffusion. Detailed of these methods present by Toro (2001) and Leveque (2002). Recently, several successful schemes have been presented to solve the SWE on unstructured cells by using finite volume formulations, Ghostine et al. (2009) and Ullrich et al. (2010).
In general, SWE have three different source terms including bed slope, bed friction and turbulence terms. Methods of calculation of above terms have important role in final results. It's more important when the flow is supercritical with standing shock waves and the domain of solution with steep slopes or having discontinuity in bed level. These conditions are often produced in steep mountain rivers. Normally the bed slope source term calculate with central method, Toro (2001) and Ghostine et al. (2009). In recent years many numerical schemes have been adopted for improving the methods of calculating the source terms; That so called "balance hyperbolic laws", Bastin et al. (2008). Different schemes are obtained according to the discretization of the source terms; for example: an upwind scheme presented by Bermudez and Vazquez (1994), the well-balanced DFB technique so based on the idea that the bed slope source term corresponds to the static force due to the non-horizontality of the bottom presented by Valiani and Begnudelli (2006) or the spatial method presented by Caleffi and Valiani (2009).
In this paper we solve SWE with finite volume Roe-TVD method in unstructured triangular cells. The multidimensional slope-limiters of Yoon and Kang (2004) are employed to achieve the second-order spatial accuracy and to prevent spurious oscillations. Manning's formula has been used to compute the bed friction and algebraic stress model (ASM), are used to calculate turbulence effects. For the inspection of the bed slope effects an ordinary central method and an upwind scheme are examined. The numerical results are compared with the experimental data to verify the accuracy of the computations.

2. The Governing Equations

Two dimensional shallow water equations with source terms can be written as:

$$\frac{\partial W}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \sum_{k=1}^{3} G_k \tag{1}$$

in which

$$W = \begin{pmatrix} h \\ q_x \\ q_y \end{pmatrix} \quad ; \quad F_x = \begin{pmatrix} q_x \\ \dfrac{q_x^2}{h} + \dfrac{gh^2}{2} \\ \dfrac{q_x q_y}{h} \end{pmatrix} \quad ; \quad F_y = \begin{pmatrix} q_y \\ \dfrac{q_x q_y}{h} \\ \dfrac{q_y^2}{h} + \dfrac{gh^2}{2} \end{pmatrix} \tag{2}$$

where W is the vector of the conserved variables, including the water depth h and the unit discharges q_x and q_y. The vectors F_x and F_y account for the convective fluxes in the x and y directions, respectively, and g is the acceleration due to gravity. The vector G_k is a source term composed of the bed slope G_1, bed friction G_2, and turbulence terms G_3 :

$$G_1 = \begin{pmatrix} 0 \\ -gh\dfrac{\partial Z_b}{\partial x} \\ -gh\dfrac{\partial Z_b}{\partial y} \end{pmatrix} ; \quad G_2 = \begin{pmatrix} 0 \\ -\dfrac{\tau_{b,x}}{\rho} \\ -\dfrac{\tau_{b,y}}{\rho} \end{pmatrix} ; \quad G_3 = \begin{pmatrix} 0 \\ -\dfrac{\partial h\overline{u_i'u_j'}}{\partial x} \\ -\dfrac{\partial h\overline{u_i'u_j'}}{\partial y} \end{pmatrix} \quad (i, j = 1,2) \tag{3}$$

Where Z_b is the bed elevation, $\tau_{b,x}$ and $\tau_{b,y}$ are the bed shear stresses due to friction in the x and y directions, ρ is the fluid density and $\overline{u_i'u_j'}$ are the depth-averaged horizontal Reynolds stresses.

3. Numerical Solution

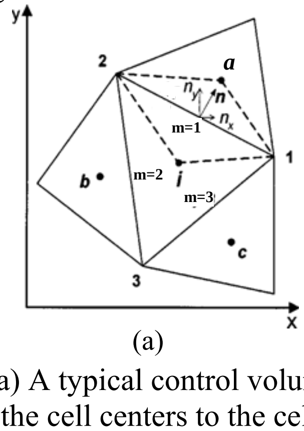
By time discretization of the system (1) and simplification, the following equations are obtained with a second-order of accuracy in time, Cea (2005):

$$\begin{aligned} W^{n+\frac{1}{2}} &= W^n - \frac{\Delta t}{2}\left(\frac{\partial F_x}{\partial x}\left(W^n\right) + \frac{\partial F_y}{\partial y}\left(W^n\right)\right) + \frac{\Delta t}{2}\sum_{k=1}^{3} G_k^n \\ W^{n+1} &= W^n - \Delta t\left(\frac{\partial F_x}{\partial x}\left(W^{n+\frac{1}{2}}\right) + \frac{\partial F_y}{\partial y}\left(W^{n+\frac{1}{2}}\right)\right) + \Delta t\sum_{k=1}^{3} G_k^{n+\frac{1}{2}} \end{aligned} \tag{4}$$

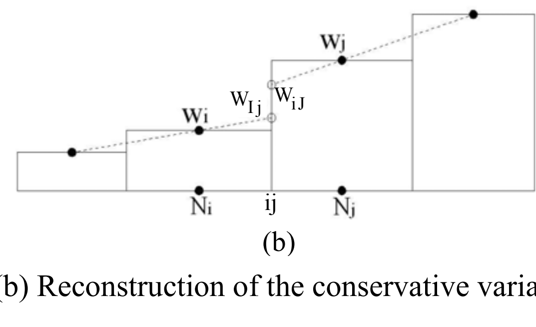
where W^n is the vector of conserved variables at time t^n and Δt is the time step. For spatial discretization, an upwind model may be implemented. For this purpose, The integration of system (1) over a cell i with area A_i and implementation of the Gauss divergence theorem lead to:

$$A_i \frac{W_i^{n+1} - W_i^n}{\Delta t} + \int_{L_i} \left(F_x\, \tilde{n}_x + F_y\, \tilde{n}_y\right) dL = \sum_{k=1}^{3} \int_{C_i} G_k\, dA \tag{5}$$

where L_i is the boundary of the cell i, and $\tilde{n} = (\tilde{n}_x, \tilde{n}_y)$ is the unit vector normal to the cell face and , Fig. 1-a.



(a)



(b)

Fig 1: (a) A typical control volume cell. (b) Reconstruction of the conservative variables from the cell centers to the cell faces

The second term on the left hand side of Eq. (5) may be calculated by Roe upwind scheme, Cea and Vazquez-Cendon (2008). To achieve a second order accuracy in the method of Roe and having TVD scheme, the conserved variables at the triangular cell faces are reconstructed using a multi dimensional limiting technique, Yoon and Kang (2004). In this method gradients of conserve variable in each cells are modified by using weighting functions, Fig. 1. The friction source term, G_2 in Eqs. (3) may be defined as, Cea (2005):

$$G_2 = \begin{pmatrix} 0 \\ -c_f U_x |U_x| \\ -c_f U_y |U_y| \end{pmatrix} \quad ; \quad c_f = \frac{gn^2}{h^{1/3}} \tag{6}$$

depth-averaged algebraic stress model used For calculating the turbulence term, G_3 in Eqs. (3). In this model, algebraic equations are used to obtain the Reynolds stresses. Details of this model is presented by Alamatian and Jafarzadeh (2011).

4. Discretization of Bed Slope Source Term

Traditionally, a centred scheme is generally used to discrete the bed slope source term, Ghostine et al. (2009). In this case G_1 in Eqs. (3) are calculated with the variable in central of cell:

$$G_{1i} = \begin{pmatrix} 0 \\ -gh_i s_{0x} \\ -gh_i s_{0y} \end{pmatrix} \tag{7}$$

Where s_{0x} and s_{0y} are cell bed slope in *x* and *y* direction and h_i is water depth in centre of cell *i*. However, Bermudez and Vazquez (1994) showed the importance of using an upwind discretization of the bed slope source term in order to avoid non-physical oscillations in the solution. they shown that if the flux is upwind the bed slope source term should also be upwind. The upwind discretization of the source terms has also been studied by other researchers such as Rebolloet al. (2003) and LeVeque (1998). One way in which the bed slope source term can be discretised in order to obtain the exact hydrostatic flow solution presented by Bermudez and Vazquez-Cendon (1994). In this way the integral of the source term over the cell C_i is divided in the sum of integrals over the sub-triangles. Each of the integrals over the sub-triangles is computed by means of a discrete source function, which gives the upwind character to the scheme. The discrete source function depends on the values of the variables at each side of the face, and on the normal vector to the face. If the second order extension of the Roe's schemes is used, the exact balance between the numerical flux and the bed slope source term is broken. To keep this balance its necessary to consider additional terms. Finally The source term is discretized as:

$$S_i = \sum_{j \in K_i} -\frac{L_{ij}}{2} g \frac{h_i + h_j}{2} \left(Z_{b,j} - Z_{b,i}\right) \begin{pmatrix} 0 \\ \tilde{n}_{x,ij} \\ \tilde{n}_{y,ij} \end{pmatrix} + \sum_{j \in K_i} \frac{L_{ij}}{2} \tilde{c}_{ij} \left(Z_{b,j} - Z_{b,i}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$G_1 = \frac{1}{A_i} \left[S_i - \sum_{j \in K_i} \left[-\frac{L_{ij}}{2} g \frac{h_i + h_{Ij}}{2} 2\left(Z_{b,Ij} - Z_{b,i}\right) \right] \begin{pmatrix} 0 \\ \tilde{n}_{x,ij} \\ \tilde{n}_{y,ij} \end{pmatrix} \right] \tag{8}$$

where K_i is the number of faces at cell *i* (triangular cells: $K_i = 3$), L_{ij} is the length of the face between cells *i* and *j* (cell face *ij*) and h_{Ij} is estimating of water depth in cell *i* at cell face *ij*.

5. Application of the Numerical Model

For the evaluation of the different method for calculation of bed slope source terms, the supercritical flow in a canal with a side-baffle is simulated. The length and width of the canal in the numerical model are 0.4 and 5.0 m respectively. The longitudinal slope of the canal is $S_{0x} = 0.00624$ and Manning roughness coefficient assume n=0.0104. A thin solid plate, as a side-baffle, is considered normal to the sidewall of the flume, 1.0 *m* downstream from the beginning of the canal. the solution domain is divided into 8305 unstructured triangular cells. A constant discharge Q=0.0372 m^3/s is introduced at the upstream boundary. At the downstream end, the dependent variables (h, q_x and q_y) are interpolated from the solution domain, which is reasonable due to the supercritical nature of the flow. The solid boundaries are simulated using the characteristics theory, Yoon and Kang (2004). The normal depth of flow at this discharge without the presence of the baffle has been 0.07 *m*, (in laboratory flume, Alamatian and Jafarzadeh (2011)). When the side-baffle is fixed, the water rose at the upstream side and a jet of flow is released to the downstream canal from the partial opening. The numerical results will be presented at three longitudinal sections, p_1, p_2 and p_3, at 0.1, 0.2 and 0.3 m from the sidewall, as shown in Fig 2.

In Fig 3 the longitudinal profiles along p_1, p_2 and p_3 are plotted using different methods for calculating the bed slope source term for side baffle width of 0.12 *m*. in this figure the numerical results compared with the experimental data.

Obviously, the numerical model is able to simulate the oblique waves downstream of the baffle successfully. The amplitude and phase of the numerical shocks are in agreement with the experimental measurements in center method. But in upwind method numerical shocks has phase lag. Based on these profiles, the central method for bed slope source terms offers the most favorable results.

In Figs 4 a-d, cross sectional profiles of the flow depth at various locations after the baffle, obtained from different models, are compared with the measured values. It may be observe that the center scheme shows a better performance than the upwind model, specifically at the standing wave fronts.

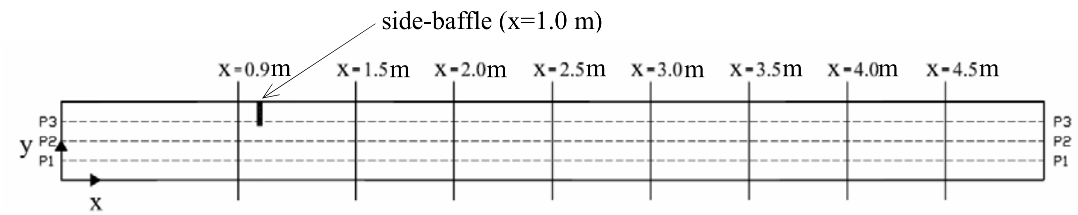
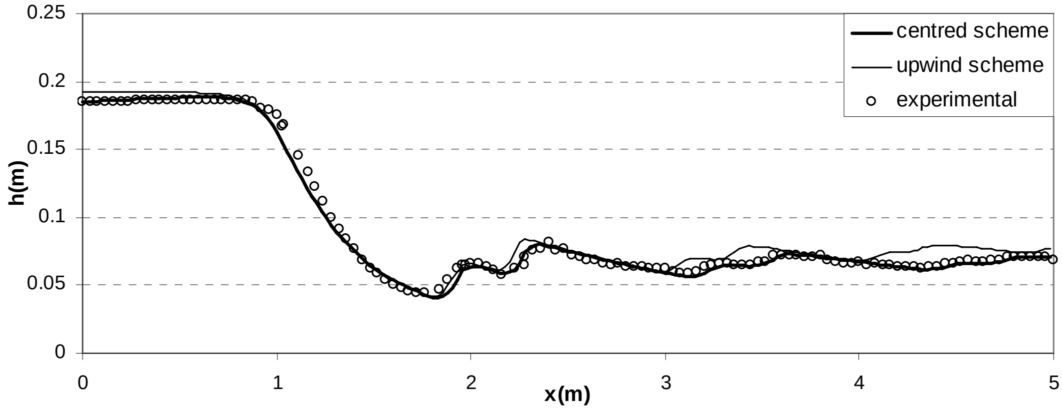


Fig 2: The longitudinal and transversal cross

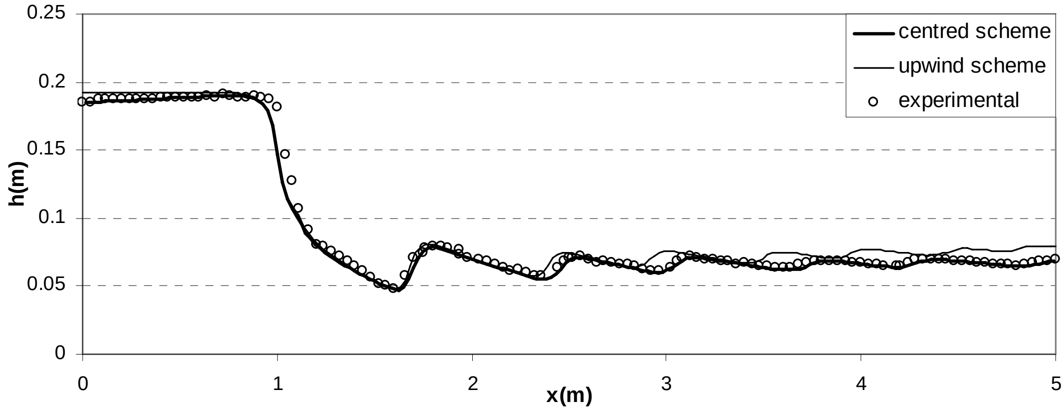
A quantitative evaluation may be obtained by comparison of the relative error norm E, defined as:

$$E = \frac{\sum |h_{num} - h_{\exp}|}{\sum |h_{\exp}|} \times 100 \qquad (9)$$

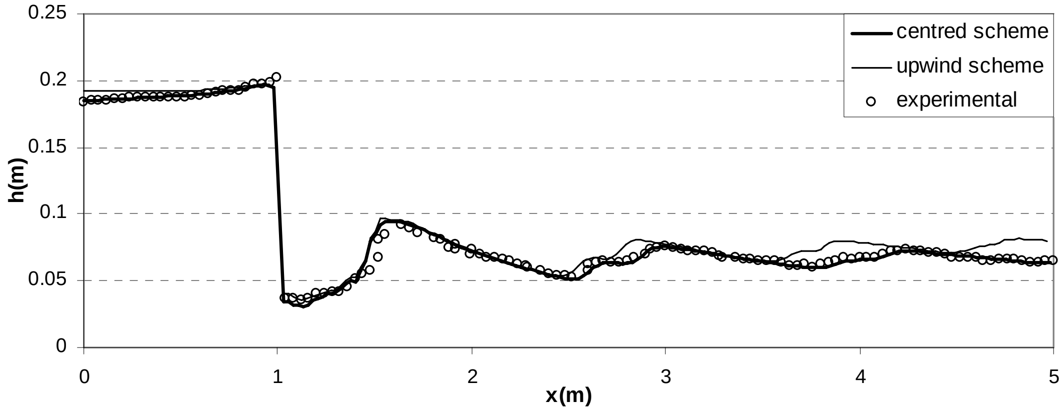
where h_{num} is the numerical flow depth, and h_{exp} is the experimental depth. The relative error norms for the two methods of calculating bed slope source term are plotted in Fig 5 for the longitudinal profiles at different baffle widths. It is clear that the center method presents the least error norms.



a. along p_1

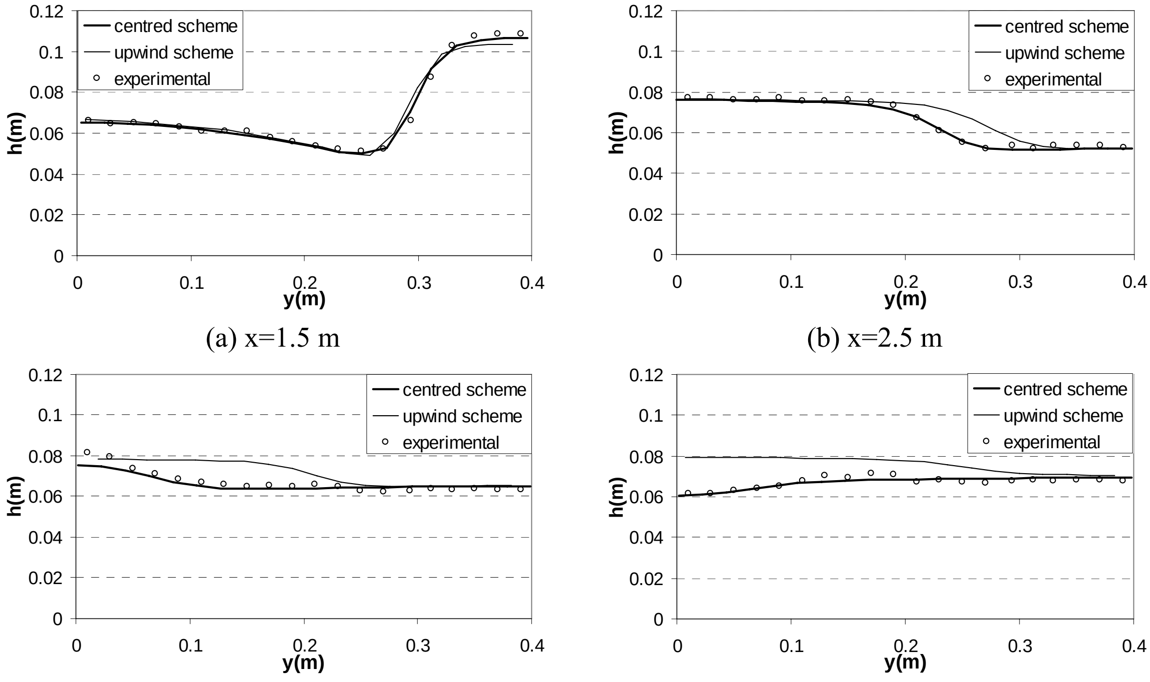


b. along p_2



c. along p_3

Fig 3: Comparison between the experimental and numerical depth profiles for a 0.12 *m* baffle width



(a) x=1.5 m (b) x=2.5 m

(c) x=3.5 m (d) x=4.5 m

Fig 4: Comparison between experimental and numerical transversal depth profiles for a 0.12 *m* side-baffle width

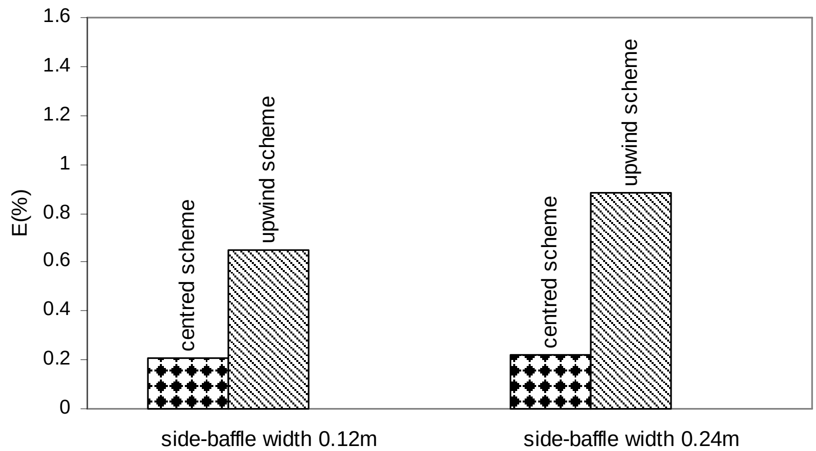


Fig 5: Error norm E along different directions for different side-baffle widths

6. Conclusions

In this article, using the 2D shallow water equations, the effects of center and upwind methods for calculating bed slope source term on the performance of standing oblique shock waves were investigated. The finite volume scheme of Roe-TVD with an unstructured triangular mesh was applied. To avoid spurious oscillations at the regions where the gradients of the variable were considerable, advanced slope limiter functions were implemented in the numerical algorithm. The effects of the bed slope, bed friction and turbulences were considered in the source terms. The bed friction term was computed using the manning formula. Turbulence source term was computed with Algebraic Stress Model(ASM). The comparison of the numerical predictions and experimental data confirmed the robustness of the numerical model. Based on our overall findings, the center method for calculating bed slope source term offered superior results to the other models. The quantitative error analysis confirmed this finding as well.

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